

# Frames for weighted shift-invariant spaces

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**Abstract.** In this paper we prove the equivalence of the frame property and the closedness for a weighted shift-invariant space

$$V_\mu^p(\Phi) = \left\{ \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} c_i(j) \phi_i(\cdot - j) \mid \{c_i(j)\}_{j \in \mathbb{Z}^d} \in \ell_\mu^p \right\}, \quad p \in [1, \infty],$$

which corresponds to  $\Phi = \Phi^r = (\phi_1, \phi_2, \dots, \phi_r)^T \in (W_\omega^1)^r$ . We, also, construct a sequence  $\Phi^{2k+1}$  and the sequence of spaces  $V_\mu^p(\Phi^{2k+1})$ ,  $k \in \mathbb{N}$ , on  $\mathbb{R}$ , with the useful properties in sampling, approximations and stability.

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**Key Words and Phrases:**  $p$ -frame; Banach frame; weighted shift-invariant space.

## 1 Introduction

In this paper, we investigate weighted shift-invariant spaces quoted in the abstract by following the methods from [2] and [25]. Such spaces figure in several areas of applied mathematics, notably in wavelet theory and approximation theory ([2], [8]). In recent years, they have been extensively studied by many authors (see [1]-[8], [14]-[16], [19], [20], [25], [26]). Sampling with non-bandlimited functions in shift-invariant spaces is a suitable and realistic model for many applications, such as modeling signals with the spectrum that is smoother than in the case of bandlimited functions, or for the numerical implementation (see [6], [9], [10], [12], [13], [17]). These requirements can often be met by choosing appropriate functions in  $\Phi$ . This means that the functions in  $\Phi$  have a shape corresponding to a particular impulse response of a device, or that they are compactly supported or that they have a Fourier transform decaying smoothly to zero as  $|\xi| \rightarrow \infty$ .

Weighted shift-invariant spaces  $V_\mu^p(\Phi)$ ,  $p \in [1, \infty]$ , where  $\mu$  is a weight, were introduced for the non uniform sampling as a direct generalization of the space  $V^p(\Phi)$  ([1], [26]). The determination of  $p$  and the signal smoothness are used for optimal compression and coding signals and images (see [9]).

The first aim of this paper is to show that the main result of [2] holds in the case of weighted shift-invariant spaces which correspond to  $L_\mu^p$  and  $\ell_\mu^p$ , i.e., weighted  $L^p$  and  $\ell^p$  spaces, respectively. Namely, we follow [2] and [25] and prove assertions which need additional arguments depending on the weights. We show that under the appropriate conditions on the frame vectors, there is an equivalence between the concept of  $p$ -frames, Banach frames with respect to  $\ell_\mu^p$  and closedness of the space which they generate. A weighted analog of Corollary 3.2 from [25] simplifies a part of the proof of our main result. Although another part of the proof follows, step by step, the proof of the corresponding theorem in [2], we think that it is not simple at all, and that it is worth to be done.

The second aim of this paper is to construct  $V_\mu^p(\Phi^{2k+1})$  spaces with specially chosen functions,  $\phi_0, \phi_1, \dots, \phi_{2k}$ , that generate a Banach frame for the shift-invariant space  $V_\mu^p(\Phi^{2k+1})$ . Actually, we take functions from a sequence  $\{\phi_i\}_{i \in \mathbb{Z}}$  so that the sequence of Fourier transforms  $\hat{\phi}_i = \theta(\cdot + i\pi)$ ,  $i \in \mathbb{Z}$ ,  $\theta \in C_0^\infty(\mathbb{R})$ , makes a partition of unity in the frequency domain ( $\mathbb{Z} = \mathbb{N}_0 \cup -\mathbb{N}$ ,  $\mathbb{N}$  is the set of natural numbers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ). We note that properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in  $V_\mu^p(\Phi)$  ([26]). Also, we note that  $\{\phi_i(\cdot - k) \mid k \in \mathbb{Z}, i = 1, \dots, r\}$  forms a Riesz basis for  $V_\mu^p(\Phi)$  when the spectrum of the Gram matrix  $[\hat{\Phi}, \hat{\Phi}](\xi)$  is bounded and bounded away from zero (see [8]). The  $d$ -dimensional case,  $d > 1$ , is technically more complicated and because of that it is not considered in this paper.

The paper is organized as follows. In Section 2 we quote basic properties of subspaces of weighted  $L^p$  and  $\ell^p$  spaces. The weighted shift-invariant spaces are investigated in Section 3, where we presented our first result quoted in the abstract, Theorem 3.10. In Section 4 we show relations between the dual of the Fréchet space  $\bigcap_{s \in \mathbb{N}_0} V_{(1+|x|^2)^{s/2}}^p(\Phi)$  and the space of periodic distributions. The case of periodic ultradistributions is obtained by using subexponential growth functions. In Section 5, we use a special sequence of functions  $\{\phi_k \mid k \in \mathbb{N}\}$  to construct a sequence of  $p$ -frames. Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust. In the final remark of Section 5, we list good properties of these frames.

## 2 Basic spaces

Denote by  $L_{loc}^1(\mathbb{R}^d)$  the space of measurable functions integrable over compact subsets of  $\mathbb{R}^d$ . For a nonnegative function  $\omega \in L_{loc}^1(\mathbb{R}^d)$  we say that is submultiplicative if  $\omega(x+y) \leq \omega(x)\omega(y)$ ,  $x, y \in \mathbb{R}^d$  and a function  $\mu$  on  $\mathbb{R}^d$  is  $\omega$ -moderate if  $\mu(x+y) \leq C\omega(x)\mu(y)$ ,  $x, y \in \mathbb{R}^d$ . We assume that  $\omega$  is continuous and symmetric and both  $\mu$  and  $\omega$  call weights, as usual. The standard class of weights on  $\mathbb{R}^d$  are of the polynomial type  $\omega_s(x) = (1 + |x|)^s$ ,  $s \geq 0$ . To quantify faster decay of functions we use the subexponential weights  $\omega(x) = e^{\alpha|x|^\beta}$ , for some  $\alpha > 0$  and  $0 < \beta < 1$ . Weighted  $L^p$  spaces with moderate weights

are translation-invariant spaces (see [1]). We, also, consider weighted sequence spaces  $\ell_\mu^p(\mathbb{Z}^d)$  with  $\omega$ -moderate weight  $\mu$ . Recall, a sequence  $c$  belongs to  $\ell_\mu^p(\mathbb{Z}^d)$  if  $c\mu$  belongs to  $\ell^p(\mathbb{Z}^d)$ .

In the sequel  $\omega$  is a submultiplicative weight, continuous and symmetric and  $\mu$  is  $\omega$ -moderate. Let  $p \in [1, \infty)$ . Then (with obvious modification for  $p = \infty$ )

$$\mathcal{L}_\omega^p = \left\{ f \mid \|f\|_{\mathcal{L}_\omega^p} = \left( \int_{[0,1]^d} \left( \sum_{j \in \mathbb{Z}^d} |f(x+j)| \omega(x+j) \right)^p dx \right)^{1/p} < +\infty \right\},$$

$$W_\omega^p := \left\{ f \mid \|f\|_{W_\omega^p} = \left( \sum_{j \in \mathbb{Z}^d} \sup_{x \in [0,1]^d} |f(x+j)|^p \omega(j)^p \right)^{1/p} < +\infty \right\}.$$

Obviously, we have  $W_\omega^p \subset W_\omega^q \subset \mathcal{L}_\omega^\infty \subset \mathcal{L}_\omega^q \subset \mathcal{L}_\omega^p \subset L_\omega^p$ ,  $W_\omega^p \subset W_\mu^p \subset W_\mu^q \subset L_\mu^q$  and  $L_\omega^p \subset L_\mu^p$ , where  $1 < p < q \leq +\infty$ . For  $p = 1$  and  $\omega = 1$  we have  $\mathcal{L}^1 = L^1$ . We also have  $\ell_\omega^1 \subset \ell_\omega^p \subset \ell_\omega^q \subset \ell_\mu^q$ , for  $1 < p < q \leq +\infty$ . From [1] we have the following properties.

- 1) If  $f \in L_\mu^p$ ,  $g \in L_\omega^1$  and  $p \in [1, \infty]$ , then  $\|f * g\|_{L_\mu^p} \leq \|f\|_{L_\mu^p} \|g\|_{L_\omega^1}$ .
- 2) If  $f \in L_\mu^p$ ,  $g \in W_\omega^1$  and  $p \in [1, \infty]$ , then  $\|f * g\|_{W_\mu^p} \leq \|f\|_{L_\mu^p} \|g\|_{W_\omega^1}$ .
- 3) If  $c \in \ell_\mu^p$  and  $d \in \ell_\omega^1$ , then holds the inequality  $\|c * d\|_{\ell_\mu^p} \leq \|c\|_{\ell_\mu^p} \|d\|_{\ell_\omega^1}$ .

Denote by  $\mathcal{WC}_\mu^p$ ,  $p \in [1, \infty]$ , a space of all  $2\pi$ -periodic functions with their sequences of Fourier coefficients in  $\ell_\mu^p$ . Let  $D_1$  and  $D_2$  be the sequences of Fourier coefficients of  $2\pi$ -periodic functions  $K_1$  and  $K_2$ , respectively. If  $D_1 * D_2 \in \ell_\mu^p$ , then  $D_1 * D_2$  is the sequence of Fourier coefficients of the product  $K_1 K_2$ . For  $K = (K_1, \dots, K_r)^T \in (WC_\mu^p)^r$ , ( $T$  means transpose) define  $\|K\|_{\ell_{\mu,*}^p}$  to be the  $\ell_\mu^p$  norm of its sequence of Fourier coefficients.

In the sequel we use the notation  $\Phi = (\phi_1, \dots, \phi_r)^T$ . Define  $\|\Phi\|_{\mathcal{H}} = \sum_{i=1}^r \|\phi_i\|_{\mathcal{H}}$ , where  $\mathcal{H} = L_\omega^p$ ,  $\mathcal{L}_\omega^p$  or  $W_\omega^p$ ,  $p \in [1, \infty]$ .

We list several lemmas needed to prove our results. Their proofs are analogous to the proof of the corresponding lemmas in [2].

**Lemma 2.1.** *Let  $f \in L_\mu^p$  and  $g \in W_\omega^1$ ,  $p \in [1, \infty]$ . Then the sequence*

$$\left\{ \int_{\mathbb{R}^d} f(x)g(x-j)dx \right\}_{j \in \mathbb{Z}^d} \in \ell_\mu^p$$

$$\text{and } \left\| \left\{ \int_{\mathbb{R}^d} f(x)g(x-j)dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^p} \leq \|f\|_{L_\mu^p} \|g\|_{W_\omega^1}.$$

Let  $c = \{c_i\}_{i \in \mathbb{N}} \in \ell_\mu^p$  and  $f \in L_\omega^p$ ,  $p \in [1, \infty]$ . We define, as in [2], their semi-convolution  $f *' c$  by  $(f *' c)(x) = \sum_{j \in \mathbb{Z}^d} c_j f(x-j)$ ,  $x \in \mathbb{R}^d$ .

- Lemma 2.2.** a) If  $f \in L_\omega^p$  and  $c \in \ell_\mu^p$ ,  $p \in [1, \infty]$ , then  $f *' c \in L_\mu^p$  and  $\|f *' c\|_{L_\mu^p} \leq \|c\|_{\ell_\mu^p} \|f\|_{L_\omega^p}$ .
- b) If  $f \in \mathcal{L}_\omega^p$ ,  $p \in [1, \infty]$ , and  $c \in \ell_\mu^1$ , then  $\|f *' c\|_{\mathcal{L}_\mu^p} \leq \|c\|_{\ell_\mu^1} \|f\|_{\mathcal{L}_\omega^p}$ .
- c) If  $f \in W_\omega^p$ ,  $p \in [1, \infty]$ , and  $c \in \ell_\mu^1$ , then  $\|f *' c\|_{W_\mu^p} \leq \|c\|_{\ell_\mu^1} \|f\|_{W_\omega^p}$ .
- d) If  $f \in W_\omega^1$  and  $c \in \ell_\mu^p$ ,  $p \in [1, \infty]$ , then  $\|f *' c\|_{W_\mu^p} \leq \|c\|_{\ell_\mu^p} \|f\|_{W_\omega^1}$ .

### 3 Characterization of $V_\mu^p(\Phi)$

In [11] Feichtinger and Gröchening extended the notation of atomic decomposition to Banach spaces ([10], [12]), while Gröchening [18] introduced a more general concept of decomposition through Banach frames. We recall the definition.

Let  $X$  be a Banach space and  $\Theta$  be an associated Banach space of scalar valued sequences, indexed by  $I = \mathbb{N}$  or  $I = \mathbb{Z}$ . Let  $\{f_n\} \subset X^*$  and  $S : \Theta \rightarrow X$  be given. The pair  $(\{f_n\}_{n \in I}, S)$  is called a Banach frame for  $E$  with respect to  $\Theta$  if

- (1)  $\{f_n(x)\}_{n \in I} \in \Theta$  for each  $x \in X$ ,
- (2) there exist positive constants  $A$  and  $B$  with  $0 < A \leq B < +\infty$  such that  $A\|x\|_X \leq \|\{f_n(x)_{n \in I}\}\|_\Theta \leq B\|x\|_X$ ,  $x \in X$ ,
- (3)  $S$  is a bounded linear operator such that  $S(\{f_n(x)\}_{n \in I}) = x$ ,  $x \in X$ .

It is said that a collection  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$  is a  $p$ -frame for  $V_\mu^p(\Phi)$  if there exists a positive constant  $C$  (depending on  $\Phi$ ,  $p$  and  $\omega$ )

$$C^{-1}\|f\|_{L_\mu^p} \leq \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \phi_i(x - j) dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^p} \leq C\|f\|_{L_\mu^p}, \quad f \in V_\mu^p(\Phi). \quad (3.1)$$

A typical application is the problem of finding a shift-invariant space model that describes a given class of signals or images (e.g. the class of chest X-rays). The observation set of  $r$  signals or images  $f_1, \dots, f_r$  may be theoretical samples, or experimental data.

Recall [1], the shift-invariant spaces are defined by

$$V_\mu^p(\Phi) := \left\{ f \in L_\mu^p \mid f(\cdot) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} c_j^i \phi_i(\cdot - j), \quad \{c_j^i\}_{j \in \mathbb{Z}^d} \in \ell_\mu^p, \quad 1 \leq i \leq r \right\}.$$

*Remark 3.1.* If  $\Phi \in W_\omega^1$  and  $\mu$  is  $\omega$ -moderate, then  $V_\mu^p(\Phi)$  is a subspace (not necessarily closed) of  $L_\mu^p$  and  $W_\mu^p$  for any  $p \in [1, \infty]$ . If  $r = 1$  and  $\{\phi(\cdot - j) \mid j \in \mathbb{Z}^d\}$  is a  $p$ -frame for  $V_\mu^p(\phi)$ , then  $V_\mu^p(\phi)$  is a closed subspace of  $L_\mu^p$  and  $W_\mu^p$  for  $p \in [1, \infty]$  (see [23]).

Let  $[\widehat{\Phi}, \widehat{\Phi}](\xi) = \left[ \sum_{k \in \mathbb{Z}^d} \widehat{\phi}_i(\xi + 2k\pi) \overline{\widehat{\phi}_j(\xi + 2k\pi)} \right]_{1 \leq i \leq r, 1 \leq j \leq r}$  where we assume that  $\widehat{\phi}_i(\xi) \overline{\widehat{\phi}_j(\xi)}$  is integrable for any  $1 \leq i, j \leq r$ . Let  $A = (a(j))_{j \in \mathbb{Z}^d}$  be an  $r \times \infty$  matrix and  $A\overline{A}^T = \left[ \sum_{j \in \mathbb{Z}^d} a_i(j) \overline{a_{i'}(j)} \right]_{1 \leq i, i' \leq r}$ . Then  $\text{rank } A = \text{rank } A\overline{A}^T$ .

Also, since  $[\widehat{\Phi}, \widehat{\Phi}](\xi)$  is continuous (as a function with  $r^2$  components) for any  $\Phi \in (\mathcal{L}_\omega^2)^r$ , it follows that  $\{\xi \in \mathbb{R}^d \mid \text{rank} [\widehat{\Phi}(\xi + 2k\pi)_{k \in \mathbb{Z}^d}] > k_0\}$  is an open set for any  $k_0 > 0$  and  $\Phi \in (\mathcal{L}_\omega^2)^r$ .

Denote by  $\Sigma_\alpha^\mu$  the family of all  $\alpha$ -slant matrices  $A = [a(j, k)]_{j \in \mathbb{Z}^d, k \in \mathbb{Z}^d}$  with

$$\|A\|_{\Sigma_\alpha^\omega} = \sum_{k \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |a(k, j)| \chi_{k+[0,1)^d}(k - \alpha j) < \infty,$$

where  $\mu$  is a weight on  $\mathbb{R}^d$  and  $\alpha$  is a positive number. The slanted matrices appear in wavelet theory, signal processing and sampling theory (see [25]). Note  $\Sigma_\alpha^\mu \subset \Sigma_\alpha^{\mu_0}$  for any weight  $\mu$  where  $\mu_0 \equiv 1$  is the trivial weight.

We assume in this subsection that  $\Phi = (\phi_1, \dots, \phi_r)^T \in (\mathcal{L}_\omega^p)^r$  for  $p \in [1, \infty)$ .

To prove Theorem 3.10, we need several lemmas. First we recall a result from [2].

**Lemma 3.2** ([2]). *The following statements are equivalent.*

- 1)  $\text{rank} [\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}^d}]$  is a constant function on  $\mathbb{R}^d$ .
- 2)  $\text{rank} [\widehat{\Phi}, \widehat{\Phi}](\xi)$  is a constant function on  $\mathbb{R}^d$ .
- 3) There exists a positive constant  $C$  independent of  $\xi$  such that

$$C^{-1} [\widehat{\Phi}, \widehat{\Phi}](\xi) \leq [\widehat{\Phi}, \widehat{\Phi}](\xi) \overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)^T} \leq C [\widehat{\Phi}, \widehat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d.$$

The proofs of the following two lemmas are similar to proofs of the corresponding lemmas from [2]; hence we will not include them here. The second one provides a localization technique in Fourier domain. It allows us to replace locally the generator  $\widehat{\Phi}$  of size  $r$  by  $\widehat{\Psi}_{1,\lambda}$  of size  $k_0$ .

**Lemma 3.3.** *All the entries of  $[\widehat{\Phi}, \widehat{\Phi}](\xi)$  belong to  $\mathcal{WC}_\omega^1$  and are continuous.*

**Lemma 3.4.** *Let the  $\text{rank} [\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}^d}] = k_0 \geq 1$  for all  $\xi \in \mathbb{R}^d$ . Then there exist a finite index set  $\Lambda$ , points  $\eta_\lambda \in [-\pi, \pi]^d$ ,  $0 \leq \delta_\lambda < 1/4$ , a nonsingular  $2\pi$ -periodic  $r \times r$  matrix  $P_\lambda(\xi)$  with all entries in the class  $\mathcal{WC}_\omega^1$  and  $K_\lambda \subset \mathbb{Z}^d$  with cardinality  $k_0$  for all  $\lambda \in \Lambda$ , such that:*

- (i)  $\bigcup_{\lambda \in \Lambda} B(\eta_\lambda, \delta_\lambda/2) \supset [-\pi, \pi]^d$ , where  $B(x_0, \delta)$  denotes the open ball in  $\mathbb{R}^d$  with center  $x_0$  and radius  $\delta$ ;

(ii)  $P_\lambda(\xi) \widehat{\Phi}(\xi) = \begin{bmatrix} \widehat{\Psi}_{1,\lambda}(\xi) \\ \widehat{\Psi}_{2,\lambda}(\xi) \end{bmatrix}$ ,  $\xi \in \mathbb{R}^d$ ,  $\lambda \in \Lambda$ , where  $\Psi_{1,\lambda}$  and  $\Psi_{2,\lambda}$  are functions from  $\mathbb{R}^d$  to  $C^{k_0}$  and  $C^{r-k_0}$ , respectively, satisfying

$$\text{rank} [\widehat{\Psi}_{1,\lambda}(\xi + 2k\pi)_{k \in K_\lambda}] = k_0, \quad \xi \in B(\eta_\lambda, 2\delta_\lambda),$$

$$\widehat{\Psi}_{2,\lambda}(\xi) = 0, \quad \xi \in B(\eta_\lambda, 8\delta_\lambda/5) + 2\pi\mathbb{Z}^d.$$

Furthermore, there exist  $2\pi$ -periodic  $C^\infty$  functions  $h_\lambda$ ,  $\lambda \in \Lambda$ , on  $\mathbb{R}^d$  such that  $\sum_{\lambda \in \Lambda} h_\lambda(\xi) = 1$ ,  $\xi \in \mathbb{R}^d$ , and  $\text{supp } h_\lambda \subset B(\eta_\lambda, \delta_\lambda/2) + 2\pi\mathbb{Z}^d$ .

The next lemma is needed for the proof of Theorem 3.10. Although the formulation is not the same as [2, Lemma 3], the proof is based on the same procedure, and we omit it.

**Lemma 3.5.** (a) Let  $\phi \in \mathcal{L}_{\omega_s}^p$  if  $p \in [1, \infty)$  and  $\phi \in W_{\omega_s}^1$  if  $p = +\infty$ . Assume that  $\sum_{j \in \mathbb{Z}^d} \phi(\cdot + j) = 0$ . Then for any function  $h$  on  $\mathbb{R}^d$  which satisfies

$$|h(x)| \leq C(1 + |x|)^{-s-d-1}, \quad |h(x) - h(y)| \leq C \frac{|x - y|}{(1 + \min\{|x|, |y|\})^{s+d+1}},$$

we have

$$\lim_{n \rightarrow +\infty} 2^{-nd} \left\| \sum_{j \in \mathbb{Z}^d} h(2^{-n}j) \phi(\cdot - j) \right\|_{\mathcal{L}_{\omega_s}^p} = 0.$$

(b) Let  $\mu(x) = e^{\alpha|x|^\beta}$ . Let  $\phi \in \mathcal{L}_\mu^p$  if  $p \in [1, \infty)$  and  $\phi \in W_\mu^1$  if  $p = +\infty$ . Assume that  $\sum_{j \in \mathbb{Z}^d} \phi(\cdot + j) = 0$ . Then for any function  $h$  on  $\mathbb{R}^d$  which satisfies

$$|h(x)| \leq C e^{-(\alpha+d+1)|x|^\beta}, \quad |h(x) - h(y)| \leq C |x - y| e^{-(\alpha+d+1)(1 + \min\{|x|^\beta, |y|^\beta\})},$$

we have

$$\lim_{n \rightarrow +\infty} 2^{-nd} \left\| \sum_{j \in \mathbb{Z}^d} h(2^{-n}j) \phi(\cdot - j) \right\|_{\mathcal{L}_\mu^p} = 0.$$

Next, we give a result on the equivalence of  $\ell_\mu^p$ -stability of the synthesis operator  $S_\Phi$  for a different  $p \in [1, \infty]$  (see [25]; here we have  $\Lambda = \{1, 2, \dots, r\}$ ).

**Proposition 3.6.** [25, Corollary 3.2] Let  $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$ ,  $p \in [1, \infty]$  and  $\mu$  is  $\omega$ -moderate. Define the synthesis operator  $S_\Phi : (\ell_\mu^p(\mathbb{Z}^d))^r \mapsto V_\mu^p(\Phi)$  by

$$S_\Phi : c = \{c_j^i\}_{j \in \mathbb{Z}^d, 1 \leq i \leq r} \mapsto \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} c_j^i \phi_i(\cdot - j).$$

If the synthesis operator  $S_\Phi$  has  $\ell_\mu^p$ -stability for some  $p \in [1, \infty]$ , i.e., there exists a positive constant  $C$  such that

$$C^{-1} \|c\|_{(\ell_\mu^p(\mathbb{Z}^d))^r} \leq \|S_\Phi c\|_{L_\mu^p} \leq C \|c\|_{(\ell_\mu^p(\mathbb{Z}^d))^r}, \quad (3.2)$$

for all  $c \in (\ell_\mu^p(\mathbb{Z}^d))^r$ , then the synthesis operator  $S_\Phi$  has  $\ell_\mu^q$ -stability for any  $q \in [1, \infty]$ .

As a consequence of the previous proposition, we have the next result.

**Proposition 3.7.** [25, Corollary 3.3] Let  $p \in [1, \infty]$  and  $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$ , and  $\mu$   $\omega$ -moderate. If the synthesis operator  $S_\Phi$  has  $\ell_\mu^p$ -stability, then there exists another family  $\Psi = (\psi_1, \dots, \psi_r)^T \in (W_\omega^1)^r$  such that the inverse of the synthesis operator  $S_\Phi$  is given by

$$(S_\Phi)^{-1}(f) = \left\{ \int_{\mathbb{R}^d} f(x) \psi_i(x-j) dx \right\}_{1 \leq i \leq r, j \in \mathbb{Z}^d}, \quad f \in V_\mu^p.$$

Proposition 3.6 and 3.7 imply:

**Theorem 3.8.** Let  $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$ ,  $p_0 \in [1, \infty]$ , and  $\mu$  is  $\omega$ -moderate. Then the following three statements are equivalent.

- a) The synthesis operator  $S_\Phi$  has  $\ell_\mu^{p_0}$ -stability.
- b)  $V_\mu^{p_0}(\Phi)$  is closed in  $L_\mu^{p_0}$ .
- c) There exists  $\Psi = (\psi_1, \dots, \psi_r)^T \in (W_\omega^1)^r$ , such that

$$f = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(\cdot - j) \rangle \phi_i(\cdot - j), \quad f \in V_\mu^{p_0}(\Phi).$$

Also we have the next assertion.

d) If the synthesis operator  $S_\Phi$  has  $\ell_\mu^{p_0}$ -stability, then the collection  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$  is a  $p_0$ -frame for  $V_\mu^{p_0}(\Phi)$ .

*Proof.* The implication  $a) \Rightarrow c)$  is a consequence of Proposition 3.6 (see Proposition 3.7).

$c) \Rightarrow a)$ : Let  $f = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(\cdot - j) \rangle \phi_i(\cdot - j)$  and

$$c^i = \{\langle f, \psi_i(\cdot - j) \rangle\}_{j \in \mathbb{Z}^d}, \quad 1 \leq i \leq r.$$

Then

$$\|c\|_{(\ell_\mu^p)^r} = \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \psi_i(x-j) dx \right\}_{j \in \mathbb{Z}^d} \right\| \leq C \|f\|_{L_\mu^{p_0}},$$

where  $C = \sum_{i=1}^r \|\psi_i\|_{W_\omega^1}$ . Using Lemma 2.1 and the inequality (2.2), we obtain the right-hand side of (3.2).

The equivalence  $a) \Leftrightarrow b)$  follows from standard functional analytic arguments (see [2, Theorem 2, Lemma 4]).

d) Lemma 2.1 implies that  $\{\langle f, \phi_i(\cdot - j) \rangle\} \in \ell_\mu^{p_0}$ ,  $1 \leq i \leq r$ , and

$$\sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \phi_i(x-j) dx \right\}_{j \in \mathbb{Z}^d} \right\| \leq \|f\|_{L_\mu^{p_0}} \sum_{i=1}^r \|\phi_i\|_{W_\omega^1}.$$

Now,  $\ell_\mu^p$ -stability implies

$$\|f\|_{L_\mu^{p_0}} \leq C \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \phi_i(x-j) dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^{p_0}}.$$

□

*Remark 3.9.* Note that  $\ell_\mu^p$ -stability of the synthesis operator implies  $\ell_\mu^q$ -stability, for any  $q \in [1, \infty]$  ([25]), so the statements b), c) and d), do not depend on  $p \in [1, \infty]$ .

Now, we give our main result.

**Theorem 3.10.** *Let  $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$ ,  $p_0 \in [1, \infty]$ , and  $\mu$  is  $\omega$ -moderate. Then the following statements are equivalent.*

- i)  $V_\mu^{p_0}(\Phi)$  is closed in  $L_\mu^{p_0}$ .
- ii)  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$  is a  $p_0$ -frame for  $V_\mu^{p_0}(\Phi)$ .
- iii) There exists a positive constant  $C$  such that

$$C^{-1}[\widehat{\Phi}, \widehat{\Phi}](\xi) \leq [\widehat{\Phi}, \widehat{\Phi}](\xi) \overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)^T} \leq C[\widehat{\Phi}, \widehat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d.$$

- iv) There exist positive constants  $C_1$  and  $C_2$  (depending on  $\Phi$  and  $\omega$ ) such that

$$C_1 \|f\|_{L_\mu^{p_0}} \leq \inf_{f = \sum_{i=1}^r \phi_i * c^i} \sum_{i=1}^r \|\{c_j^i\}_{j \in \mathbb{Z}^d}\|_{\ell_\mu^{p_0}} \leq C_2 \|f\|_{L_\mu^{p_0}}, \quad f \in V_\mu^{p_0}(\Phi). \quad (3.3)$$

- v) There exists  $\Psi = (\psi_1, \dots, \psi_r)^T \in (W_\omega^1)^r$ , such that

$$f = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(\cdot - j) \rangle \phi_i(\cdot - j) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \phi_i(\cdot - j) \rangle \psi_i(\cdot - j), \quad f \in V_\mu^{p_0}(\Phi).$$

*Proof.* If the synthesis operator has  $\ell_\mu^p$ -stability, then the statement iv) is satisfied. Conversely, if the statement iv) is satisfied, then the right-hand side of (3.2) (with  $p = p_0$ ) immediately follows. Using c) from Theorem 3.8, we obtain the left-hand side of (3.2). Hence, by Theorem 3.8, we have i)  $\Leftrightarrow$  iv) and iv)  $\Rightarrow$  ii). The equivalence iv)  $\Leftrightarrow$  v) follows from Lemma 2.1.

We follow [2] to prove iii)  $\Rightarrow$  v) and ii)  $\Rightarrow$  iii), and carefully check the use of weights.

$$\text{iii) } \Rightarrow \text{v). Let } B_\lambda(\xi) = H_\lambda(\xi) \overline{P_\lambda(\xi)^T} \begin{pmatrix} [\widehat{\Psi}_{1,\lambda}, \widehat{\Psi}_{1,\lambda}](\xi)^{-1} & 0 \\ 0 & \mathbf{I} \end{pmatrix} P_\lambda(\xi),$$

for  $h_\lambda(\xi)$ ,  $P_\lambda(\xi)$  and  $\widehat{\Psi}_{1,\lambda}$  as in Lemma 3.4. We have  $B_\lambda(\xi) \in \mathcal{WC}_\omega^p$ , for all



$p \in [1, +\infty]$ . Define  $\widehat{\Psi}(\xi) = \sum_{\lambda \in \Lambda} h_\lambda(\xi) B_\lambda(\xi) \widehat{\Phi}(\xi)$ . One has  $\Psi \in W_\omega^1$ . For any  $f \in V_\mu^p(\Phi)$ , define  $g(x) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(x-j) \rangle \phi_i(x-j)$ ,  $x \in \mathbb{R}^d$ . Since  $f \in V_\mu^p(\Phi)$ , there exists a  $2\pi$ -periodic distribution  $A(\xi) \in \mathcal{WC}_\mu^p$  such that  $\widehat{f}(\xi) = A(\xi)^T \widehat{\Phi}(\xi)$ . By Lemma 3.4, we have  $\widehat{g}(\xi) = \widehat{f}(\xi)$ .

Since  $\widehat{\Psi}(\xi) = \sum_{\lambda \in \Lambda} h_\lambda(\xi) B_\lambda(\xi) \widehat{\Phi}(\xi)$ , for  $f = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \phi_i(\cdot-j) \rangle \psi(\cdot-j)$  the proof is similar.

$ii) \Rightarrow iii)$ . Let  $k_0 = \min_{\xi \in \mathbb{R}^d} \text{rank} [\widehat{\Phi}(\xi + 2k\pi)_{k \in \mathbb{Z}^d}]$  and let

$$\Omega_{k_0} = \{\xi \in \mathbb{R}^d \mid \text{rank} [\widehat{\Phi}(\xi + 2k\pi)_{k \in \mathbb{Z}^d}] > k_0\}.$$

Then  $\Omega_{k_0} \neq \mathbb{R}^d$ . It is sufficient to prove that  $\Omega_{k_0} = \emptyset$  (see Lemma 3.2). Suppose that  $\Omega_{k_0} \neq \emptyset$ . Since  $\Omega_{k_0}$  is open set, then  $\partial\Omega_{k_0} \neq \emptyset$  and  $\text{rank} [\widehat{\Phi}(\xi_0 + 2k\pi)]_{k \in \mathbb{Z}^d} = k_0$ , for any  $\xi_0 \in \partial\Omega_{k_0}$ , and  $\max_{\xi \in B(\xi_0, \delta)} \text{rank} [\widehat{\Phi}(\xi + 2k\pi)]_{k \in \mathbb{Z}^d} > k_0$ ,  $\delta > 0$ . By Lemma 3.4, there exist a nonsingular  $2\pi$ -periodic  $r \times r$  matrix  $P_{\xi_0}(\xi)$  with all entries in the class  $\mathcal{WC}_\omega^1$ ,  $\delta_0 > 0$  and  $K_0 \subset \mathbb{Z}^d$  with cardinality  $k_0$ . Define  $\Psi_{\xi_0}$ ,  $\widehat{\Psi}_{\xi_0}(\xi)$  as in Lemma 3.4. The construction of  $\Psi_{\xi_0}$  and (2.2) imply  $\Psi_{\xi_0} \in W_\omega^1$ . Choose  $n_0$  such that  $2^{-n_0} < \delta_0$  and define  $\alpha_n(\xi)$ ,  $H_{n, \xi_0}(\xi)$  and  $\widetilde{H}_{n, \xi_0}(\xi)$  as in [2]. For any  $2\pi$ -periodic distribution  $F \in \mathcal{WC}_\mu^{p_0}$  define,  $g_n$ , for  $n \geq n_0 + 1$ , as in [2]. Note that  $g_n \in V_\mu^{p_0}(\Phi)$  and  $[\widehat{g}_n, \widehat{\Psi}_{1, \xi_0}](\xi) = 0$ . This leads to

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell_{\mu, *}^{p_0}} \leq C \|g_n\|_{L_\mu^{p_0}} \|\mathcal{F}^{-1}(H_{n, \xi_0}(\xi) \widehat{\Psi}_{2, \xi_0}(\xi))\|_{\mathcal{L}^\infty}.$$

Using Lemma 3.5, we obtain  $\lim_{n \rightarrow +\infty} \|\mathcal{F}^{-1}(H_{n, \xi_0}(\xi) \widehat{\Psi}_{2, \xi_0}(\xi))\|_{\mathcal{L}^\infty} = 0$ . There exists a sequence  $\rho_n$ ,  $n \geq n_0$ , such that  $\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell_{\mu, *}^{p_0}} \leq \rho_n \|g_n\|_{L_\mu^{p_0}}$  and  $\lim_{n \rightarrow +\infty} \rho_n = 0$ . This, together with the assumption  $ii)$  and

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell_{\mu, *}^{p_0}} = \left\| \left\{ \int_{\mathbb{R}^d} g_n(\xi) \overline{\widehat{\Phi}(\xi-j)} dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^{p_0}} \geq C \|g_n\|_{L_\mu^{p_0}},$$

leads to  $g_n = 0$ ,  $n \geq n_0 + 1$ . Then

$$\widetilde{H}_{n, \xi_0}(\xi) [\widehat{\Psi}_{1, \xi_0}, \widehat{\Psi}_{1, \xi_0}](\xi) (\alpha_n(\xi))^{-1} \widehat{\Psi}_{1, \xi_0}(\xi) = \widetilde{H}_{n, \xi_0}(\xi) \widehat{\Psi}_{2, \xi_0}(\xi), \quad (3.4)$$

for any  $2\pi$ -periodic distribution  $F \in \mathcal{WC}_\mu^{p_0}$  and  $n \geq n_0 + 1$ . We, also, get

$$\widetilde{H}_{n, \xi_0}(\xi) [\widehat{\Psi}_{1, \xi_0}, \widehat{\Psi}_{1, \xi_0}](\xi) (\alpha_n(\xi))^{-1} \widehat{\Psi}_{1, \xi_0}(\xi) = 0, \quad \xi \in B(\xi_0, 2^{-n_0-1}) + 2\pi\mathbb{Z}^d.$$

So, from (3.4) and the fact that it is valid for all  $n \geq n_0 + 1$ , we have  $\widehat{\Psi}_{2, \xi_0}(\xi) = 0$ ,  $\xi \in B(\xi_0, 2^{-n_0-3}) + 2\pi\mathbb{Z}^d$ . This contradicts the fact that  $\widehat{\Psi}_{2, \xi_0}(\xi) \neq 0$ ,  $\forall \xi \in B(\xi_0, \delta) + 2\pi\mathbb{Z}^d$ ,  $0 < \delta < 2\delta_0$ .

With this we complete the proof  $ii) \Rightarrow iii)$  and the proof of the theorem.  $\square$

*Remark 3.11.* Note that conditions in Theorem 3.8 and Theorem 3.10 do not depend on  $p \in [1, \infty]$ , so we obtain the next corollary.

**Corollary 3.12.** *Let  $\Phi \in (W_\omega^1)^r$  and  $p_0 \in [1, \infty]$ .*

- i) If  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$  is a  $p_0$ -frame for  $V_\mu^{p_0}(\Phi)$ , then  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$  is a  $p$ -frame for  $V_\mu^p(\Phi)$ , for any  $p \in [1, \infty]$ .*
- ii) If  $V_\mu^{p_0}(\Phi)$  is closed in  $L_\mu^{p_0}$  and  $W_\mu^{p_0}$ , then  $V_\mu^p(\Phi)$  is closed in  $L_\mu^p$  and  $W_\mu^p$ , for any  $p \in [1, \infty]$ .*

*Remark 3.13.* (v)  $\Rightarrow$  (ii) implies that  $\{\psi_i(\cdot - j) \mid 1 \leq i \leq r, j \in \mathbb{Z}^d\}$  is a dual  $p$ -frame of  $\{\phi_i(\cdot - j) \mid 1 \leq i \leq r, j \in \mathbb{Z}^d\}$ . So, the  $p$ -frame for  $V_\mu^p(\Phi)$  is a Banach frame (with respect to  $\ell_\mu^p$ ).

## 4 Connections with periodic distributions

We will use the notation  $V_s^p$  instead of  $V_{(1+|x|^2)^{s/2}}^p$  (similarly for  $\ell_s^p$ ). Since  $\ell_s^p$  and  $V_s^p$  are isomorphic Banach spaces for all  $s \geq 0$  and  $p \in [1, \infty]$ , we have  $V_{s_1}^p(\Phi) \subset V_{s_2}^p(\Phi)$  for  $0 \leq s_2 \leq s_1$ ,  $p \in [1, \infty]$ . We define Fréchet spaces  $X_{F,p}$ ,  $p \in [1, \infty]$ , as  $X_{F,p} = \bigcap_{s \in \mathbb{N}_0} V_s^p(\Phi)$ . Clearly,  $X_{F,p}$  is dense in  $V_s^p(\Phi)$  for all  $s \in \mathbb{N}_0$ .

The corresponding sequence space is  $Q_{F,p} = \bigcap_{s \in \mathbb{N}_0} \ell_s^p$ ,  $p \in [1, \infty]$ , which is the space of rapidly decreasing sequences  $s$ . By Corollary 3.12 it follows that the definition of  $X_{F,p}$  does not depend on  $p \in [1, \infty]$ . So we use notation  $X_F$ ,  $Q_F$  instead of  $X_{F,p}$ ,  $Q_{F,p}$ . The set  $\{\Phi(\cdot - k) \mid k \in \mathbb{Z}^d\}$  forms a  $F$ -frame for  $X_F$  since it forms a Banach frame for every space in the intersection (see [23] for the definition).

Since the corresponding function space for  $s$  is the space of rapidly decreasing functions  $\mathcal{S} = \{f \mid \|f\|_m = \sup_{n \leq m} (1 + |x|^2)^{m/2} |f^{(n)}(x)| < +\infty\}$ , and its dual is

$\mathcal{S}'$ - the space of tempered distributions, we obtain that the dual space  $X_F'$  is isomorphic to (a complemented subspace of)  $\mathcal{S}'$ .

Denote by  $\mathcal{P}(-\pi, \pi)$  the space of smooth  $2\pi$ -periodic functions on  $\mathbb{R}^d$  with the family of norms  $|\theta|_k = \sup\{|\theta^{(k)}(t)|; t \in (-\pi, \pi)\}$ ,  $k \in \mathbb{N}_0$ . It is a Fréchet space and its dual is the space of  $2\pi$ -periodic tempered distributions. We say that  $T$  is a  $2\pi$ -periodic distribution if it is a tempered distribution on  $\mathbb{R}^d$  and  $T = T(\cdot + 2j\pi)$ , for all  $j \in \mathbb{Z}^d$ . Denote by  $\mathcal{P}'(-\pi, \pi)$  the space of periodic tempered distributions (see [24]). Recall that  $\mathcal{F}(h) = \hat{h} = \int_{\mathbb{R}^d} e^{-2\pi\sqrt{-1}t} h(t) dt$  for  $h \in L^1$ .

**Theorem 4.1.** *Let  $\Phi = (\phi_1, \dots, \phi_r)^T \in \bigcap_{s \geq 0} (W_s^1)^r$  and  $\Psi = (\psi_1, \dots, \psi_r)^T$  be its dual frame (according to v) of Theorem 3.10). Then*

$$X_F = \mathcal{F}^{-1} \left( \sum_{i=1}^r \hat{\phi}_i \cdot \mathcal{P}(-\pi, \pi) \right), \quad X_F' = \mathcal{F}^{-1} \left( \sum_{i=1}^r \hat{\psi}_i \cdot \mathcal{P}'(-\pi, \pi) \right)$$

in the topological sense. Let

$$f = \sum_{k=1}^r \sum_{p \in \mathbb{Z}^d} c_p^k \phi_k(\cdot - p) \in X_F \text{ and } F = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} d_j^i \psi_i(\cdot - j) \in X'_F.$$

The dual pairing is given by

$$\langle F, f \rangle = \sum_{i=1}^r \sum_{k=1}^r \left\langle \widehat{\psi}_i(\xi) \widehat{\phi}_k(-\xi) \sum_{j \in \mathbb{Z}^d} d_j^i e^{2\pi j \xi \sqrt{-1}}, \sum_{p \in \mathbb{Z}^d} c_p^k e^{-2\pi p \xi \sqrt{-1}} \right\rangle, \quad (4.1)$$

$$\text{where } f = \sum_{k=1}^r \sum_{p \in \mathbb{Z}^d} c_p^k \phi_k(\cdot - p) \in X_F \text{ and } F = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} d_j^i \psi_i(\cdot - j) \in X'_F.$$

$$\text{In particular, we have } \int_{\mathbb{R}^d} \varphi_i \psi_k dt = \int_{\mathbb{R}^d} \widehat{\varphi}_i(\xi) \widehat{\psi}_k(-\xi) d\xi = \delta_{ik}, \quad 1 \leq i, k \leq r.$$

*Proof.* Since  $\sum_{p \in \mathbb{Z}^d} c_p^k e^{2\pi \sqrt{-1} p \xi} \in \mathcal{P}(-\pi, \pi)$ , we obtain the structure of  $f \in X_F$  as in the theorem. The same explanation works for  $X'_F$ .

By the fact that  $\langle F(x), f(x) \rangle = \langle \widehat{F}(\xi), \widehat{f}(-\xi) \rangle$ , we have that (4.1) follows.

Let  $d_0^i = \delta_{ik}$ ,  $i = 1, \dots, r$ , and  $d_j^i = 0$ ,  $j \neq 0$ , and, also, let  $c_0^k = \delta_{ik}$  for  $k = 1, \dots, r$  and  $c_p^k = 0$ ,  $p \neq 0$ . Using that, we obtain

$$\langle F(\xi), f(\xi) \rangle = \sum_{i=1}^r \sum_{k=1}^r \langle \widehat{\psi}_i(\xi), \widehat{\phi}_k(-\xi) d_0^i, c_0^k \rangle = \int_{\mathbb{R}^d} \widehat{\psi}_{k_0}(\xi) \widehat{\phi}_{k_0}(-\xi) d\xi, \quad 1 \leq k_0 \leq r.$$

On the other hand  $f(x) = \langle f(x), \psi_{k_0}(x) \rangle \phi_{k_0}(x)$  and  $f = \phi_{k_0}$  for some  $1 \leq k_0 \leq r$ , so we obtain  $\langle f, \psi_{k_0} \rangle = 1$ . Since  $F = \psi_{k_0}$ , we get  $\langle F, f \rangle = \langle f, \psi_{k_0} \rangle = 1$ . Finally, we have  $\int_{\mathbb{R}^d} \widehat{\varphi}_i(\xi) \widehat{\psi}_k(-\xi) d\xi = \delta_{ik}$ ,  $1 \leq i, k \leq r$ .  $\square$

Let  $\beta \in (0, 1)$ . Now, we consider weights  $\mu_k = e^{k|x|^\beta}$ ,  $k \in \mathbb{N}$ , and the corresponding spaces  $V_{\mu_k}^p(\Phi)$  and their intersection  $X_{F,p}^{(\beta)} = \bigcap_{k \in \mathbb{N}} V_{\mu_k}^p(\Phi)$ . It is a Fréchet space not depending on  $p$ , so we use notation  $X_F^{(\beta)}$ . The corresponding sequence space is  $s^{(\beta)} = \bigcap_{k \in \mathbb{N}} \ell_{\mu_k}^p$ , i.e., the space of subexponentially rapidly decreasing sequences determining the space of periodic tempered ultradistributions via the mapping  $s^{(\beta)} \ni (a_j)_{j \in \mathbb{Z}^d} \leftrightarrow \sum_{j \in \mathbb{Z}^d} a_j e^{j \xi \sqrt{-1}} \in \mathcal{P}(-\pi, \pi)$  (see [22]).

## 5 Construction of $p$ -frames

Let  $\theta$  be a smooth non negative function such that  $\theta(x) = 1$ ,  $x \in [-\pi + \varepsilon, \pi - \varepsilon]$ , for  $0 < \varepsilon < \frac{1}{4}$ , and  $\text{supp } \theta \subseteq [-\pi, \pi]$ . Let  $\phi_k(x) = \mathcal{F}^{-1}(\theta(\cdot + k\pi))(x)$ ,  $x \in \mathbb{R}$ ,

$k \in \mathbb{Z}$ . We can divide every  $\theta(\cdot + k\pi)$  with the sum  $\sum_{k \in \mathbb{Z}} \theta(\cdot + k\pi)$  in order to obtain the partition of unity. By the Paley-Wiener theorem, we have that  $\phi_k \in W_\mu^1(\mathbb{R})$ ,  $k \in \mathbb{Z}$ . We say that set  $\{\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_r}\}$ ,  $i_1 < i_2 < \dots < i_r$ , is a set of  $r$  successive functions if  $i_n = i_1 + (n-1)$ ,  $n = 2, \dots, r$ . Note that for every  $\xi \in \mathbb{R}$  there exist  $\xi_0 \in (-\pi, \pi)$  and  $k \in \mathbb{Z}$  such that  $\xi = \xi_0 + k\pi$ .

Now, we consider the following three cases.

1° The case of two successive functions.

If  $\Phi = (\phi_i, \phi_{i+1})^T$ ,  $i \in \mathbb{Z}$ , then  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$ ,  $\xi \in \mathbb{R}$ , is not a constant function on  $\mathbb{R}$ . In this case, for the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$ , we obtain the  $2 \times \infty$  matrix

$$A(\xi_0) = \begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots \end{bmatrix},$$

which depends on  $\xi_0 \in (-\pi, \pi)$ , where  $\alpha_{-1}^{\xi_0} = \theta(\xi_0 - \pi)$ ,  $\alpha_0^{\xi_0} = \theta(\xi_0)$  and  $\alpha_1^{\xi_0} = \theta(\xi_0 + \pi)$ .

For  $\xi_0^1 = \frac{\pi}{2}$ , we have  $\alpha_0^{\xi_0^1} \neq 0$ ,  $\alpha_{-1}^{\xi_0^1} \neq 0$ , and for  $\xi_0^2 = -\frac{\pi}{2}$ , we have  $\alpha_0^{\xi_0^2} \neq 0$ ,  $\alpha_1^{\xi_0^2} \neq 0$ . Since  $\text{rank} A(\xi_0^1) = 1$  and  $\text{rank} A(\xi_0^2) = 2$ , we conclude that for successive functions  $\phi_i, \phi_{i+1}$ ,  $i \in \mathbb{Z}$ , the rank of the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is not a constant function on  $\mathbb{R}$ .

2° The case of three successive functions.

If  $\Phi = (\phi_i, \phi_{i+1}, \phi_{i+2})^T$ ,  $i \in \mathbb{Z}$ , then  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is a constant function on  $\mathbb{R}$ . We have that  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}] = 2$ , for all  $\xi \in \mathbb{R}$ .

Indeed, the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$ ,  $\xi \in \mathbb{R}$ , is  $3 \times \infty$  matrix

$$B(\xi_0) = \begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & \cdots \end{bmatrix},$$

which depends on  $\xi_0 \in (-\pi, \pi)$ , where  $\alpha_{-1}^{\xi_0} = \theta(\xi_0 - \pi)$ ,  $\alpha_0^{\xi_0} = \theta(\xi_0)$  and  $\alpha_1^{\xi_0} = \theta(\xi_0 + \pi)$ . Since,  $\theta(\xi_0) \neq 0$  for all  $\xi_0 \in (-\pi, \pi)$ , the matrix  $B(\xi_0)$  has 2 columns with non-zero elements for all  $\xi_0 \in (-\pi, \pi)$ . So,  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is a constant function on  $\mathbb{R}$  and  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}] = 2$ , for all  $\xi \in \mathbb{R}$ .

3° The case of  $r > 3$  successive functions.

By taking  $r + 1$  successive functions  $\phi_i, \phi_{i+1}, \dots, \phi_{i+r}$ ,  $r > 2$ , we have different situations described in the next lemma.

**Lemma 5.1.** *a) If  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$ , for  $i \in \mathbb{Z}$ ,  $r \in 2\mathbb{N} + 1$ , then  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is not a constant function on  $\mathbb{R}$ .*

*b) If  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$ ,  $i \in \mathbb{Z}$ ,  $r \in 2\mathbb{N}$ , then  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is a constant function on  $\mathbb{R}$  and we have, for all  $\xi \in \mathbb{R}$ , and  $r = 2n$ ,  $n \in \mathbb{N}$ ,  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}] = n + 1$ .*

*Proof.* Since supports of products  $\widehat{\phi}_{i_1}(\xi + 2j_1\pi)\widehat{\phi}_{i_2}(\xi + 2j_2\pi)$  are non-empty if the arguments are of the form  $\xi - \pi, \xi, \xi + \pi$ , modulo  $2j\pi$ ,  $j \in \mathbb{Z}$ , we have that

only blocks with elements

$$\begin{bmatrix} \theta(\xi) & \theta(\xi + 2\pi) \\ \theta(\xi - \pi) & \theta(\xi + \pi) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \theta(\xi - \pi) & \theta(\xi + \pi) \\ \theta(\xi - 2\pi) & \theta(\xi) \end{bmatrix},$$

can determine the rank of the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$ . For any other choice of  $2 \times 2$  matrix, we get determinant equal 0.

(a) Let  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+(2n-1)})^T$ ,  $n \in \mathbb{N}$ .

For the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$ , we obtain the  $r \times \infty$  matrix

$$A_r(\xi_0) = \begin{bmatrix} \cdots & \alpha_0^{\xi_0} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_0^{\xi_0} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & \cdots \end{bmatrix},$$

where  $\alpha_{-1}^{\xi_0} = \theta(\xi_0 - \pi)$ ,  $\alpha_0^{\xi_0} = \theta(\xi_0)$  and  $\alpha_1^{\xi_0} = \theta(\xi_0 + \pi)$ ,  $\xi_0 \in (-\pi, \pi)$ .

For  $\xi_0^1 = \frac{\pi}{2}$ , we have  $\alpha_0^{\xi_0^1} \neq 0$ ,  $\alpha_{-1}^{\xi_0^1} \neq 0$ , and for  $\xi_0^2 = -\frac{\pi}{2}$ , we obtain  $\alpha_0^{\xi_0^2} \neq 0$ ,  $\alpha_1^{\xi_0^2} \neq 0$ . Since  $\text{rank } A_r(\xi_0^1) = n$  and  $\text{rank } A_r(\xi_0^2) = n + 1$ , we conclude that for even number of successive functions  $\phi_i, \phi_{i+1}, \dots, \phi_{i+(2n-1)}$ ,  $i \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , the rank of the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is not a constant function on  $\mathbb{R}$ .

(b) Let  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+2n})^T$ ,  $i \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ . The matrix

$$[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}] = \begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_0^{\xi_0} & \cdots \end{bmatrix},$$

has the constant rank on  $\mathbb{R}$ . Indeed, since  $\alpha_0^{\xi_0} \neq 0$  for all  $\xi_0 \in (-\pi, \pi)$ , the matrix  $[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  has  $n + 1$  columns with non-zero elements for all  $\xi \in \mathbb{R}$  and  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}] = n + 1$ , for all  $\xi \in \mathbb{R}$ .  $\square$

As a consequence of Corollary 3.10 and Lemma 5.1, 1° we have the next result.

**Theorem 5.2.** *Let  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+2n})^T$ , for  $i \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ . Then  $V_\mu^p(\Phi)$  is closed in  $L_\mu^p$ , for any  $p \in [1, \infty]$ , and  $\{\phi_{i+s}(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq s \leq 2n\}$  is a  $p$ -frame for  $V_\mu^p(\Phi)$  for any  $p \in [1, \infty]$ .*

*Remark 5.3.* In this way we obtain the sequence of closed spaces  $V_\mu^p(\phi_0, \phi_1, \phi_2)$ ,  $V_\mu^p(\phi_0, \phi_1, \phi_2, \phi_3, \phi_4)$ ,  $V_\mu^p(\phi_0, \phi_2, \dots, \phi_6)$ , etc. We also conclude that spaces generated with even numbers of successive functions, for example  $V_\mu^p(\phi_0, \phi_1)$ ,  $V_\mu^p(\phi_0, \phi_1, \dots, \phi_5)$ , are not closed subspaces of  $L_\mu^p$ .

**Theorem 5.4.** *Let  $\Phi = (\phi_{k_1}, \phi_{k_2}, \dots, \phi_{k_r})^T$ ,  $k_1 < k_2 < \dots < k_r$ ,  $r \in \mathbb{N}$ ,  $k_1, k_2, \dots, k_r \in \mathbb{Z}$ , and  $V_{\mu, k_1, k_2, \dots, k_r}^p = V_\mu^p(\Phi)$ . We consider the following cases.*

- i)  $k_{i+1} - k_i > 1$ ,  $i = 1, \dots, r-1$ ;
- ii) *If for some  $i_0 \in \{1, 2, \dots, r\}$  holds  $k_{i_0+1} - k_{i_0} = 1$ , then there exists  $n \in \mathbb{N}$ ,  $2 \leq 2n \leq r$ , such that  $k_{i_0} + 2, k_{i_0} + 3, \dots, k_{i_0} + 2n$  are elements of the set  $\{k_1, \dots, k_r\}$ .*

*In these cases the following statements hold.*

- 1°  $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j \in \mathbb{Z}}]$  is a constant function for all  $\xi \in \mathbb{R}$ .
- 2°  $V_\mu^p(\Phi)$  is closed in  $L_\mu^p$  for any  $p \in [1, \infty]$ .
- 3°  $\{\phi_{k_i}(\cdot - j) \mid j \in \mathbb{Z}, 1 \leq i \leq r\}$  is a  $p$ -frame for  $V_\mu^p(\Phi)$  for any  $p \in [1, \infty]$ .

*Remark 5.5.* (1) We refer to [4] and [26] for the  $\gamma$ -dense set  $X = \{x_j \mid j \in J\}$ . Let  $\phi_k(x) = \mathcal{F}^{-1}(\theta(\cdot - k\pi))(x)$ ,  $x \in \mathbb{R}$ . Following the notation of [26], we put  $\psi_{x_j} = \phi_{x_j}$  where  $\{x_j \mid j \in J\}$  is  $\gamma$ -dense set determined by  $f \in V^2(\phi) = V^2(\mathcal{F}^{-1}(\theta))$ . Checking the proofs of Theorems 3.1, 3.2 and 4.1 in [26], we obtain the same conclusions as in these theorems. These theorems show the conditions and explicit  $C_p$  and  $c_p$  such that the inequality

$$c_p \|f\|_{L_\mu^p} \leq \left( \sum_{j \in J} |\langle f, \psi_{x_j} \rangle \mu(x_j)|^p \right)^{1/p} \leq C_p \|f\|_{L_\mu^p}$$

holds. This inequality guarantee the feasibility of a stable and continuous reconstruction algorithm in the signal spaces  $V_\mu^p(\Phi)$ .

(2) Since the spectrum of the Gram matrix  $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ , for  $\Phi$  defined in Theorem 5.4, is bounded and bounded away from zero (see [8]), then the family  $\{\Phi(\cdot - j) \mid j \in \mathbb{Z}\}$  forms a  $p$ -Riesz basis for  $V_\mu^p(\Phi)$ .

(3) For the appropriate choice of function  $\Phi$ , for example  $\Phi$  defined in Theorem 5.4, the associated Gram matrix satisfies a suitable Muckenhoupt  $A_2$  condition (see [21]), so the system  $\{\Phi(\cdot - j) \mid j \in \mathbb{Z}\}$  is stable in  $L_\mu^2(\mathbb{R})$ .

(4) Frames of the above type may be useful in applications since they satisfy assumptions of Theorem 3.1 and Theorem 3.2 in [5]. They show that error analysis for sampling and reconstruction can be tolerated, or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions  $\phi_{k_1}, \dots, \phi_{k_r}$ .

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